

TWO-STEP DARBOUX TRANSFORMATIONS AND EXCEPTIONAL LAGUERRE POLYNOMIALS

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ABSTRACT. It has been recently discovered that exceptional families of Sturm-Liouville orthogonal polynomials exist, that generalize in some sense the classical polynomials of Hermite, Laguerre and Jacobi. In this paper we show how new families of exceptional orthogonal polynomials can be constructed by means of multiple-step algebraic Darboux transformations. The construction is illustrated with an example of a 2-step Darboux transformation of the classical Laguerre polynomials, which gives rise to a new orthogonal polynomial system indexed by two integer parameters. For particular values of these parameters, the classical Laguerre and the type II X_ℓ -Laguerre polynomials are recovered.

1. INTRODUCTION

It has recently been shown that the hypotheses of Bochner's Theorem [2] on the characterization of orthogonal polynomial systems defined by Sturm-Liouville problems [17] can be relaxed to give rise to new complete orthogonal polynomial systems. In some recent papers, the concept of an *exceptional polynomial subspace* [7, 8] and the closely related notion of *exceptional orthogonal polynomials* [11, 12] have been introduced. Like the classical examples, exceptional orthogonal polynomials are eigenfunctions of a second-order differential operator, but the sequence of eigenfunctions need not contain polynomials of all degrees, even though the full set of eigenfunctions still forms a basis of the weighted L^2 space.

The first examples of such polynomials were the X_1 -Jacobi and X_1 -Laguerre polynomials introduced in [11, 12]. In this co-dimension 1 case, a full characterization of all Sturm-Liouville polynomial systems is available thanks to the classification of codimension one exceptional polynomial subspaces performed in [11]. The first relation between exceptional polynomials and the Darboux transformation was made by Quesne in the context of shape invariant potentials in supersymmetric quantum mechanics [19, 20], where examples of X_2 polynomials are also found. Higher codimension families were described for the first time by Sasaki and Odake [21] for arbitrary codimension ℓ . These authors also realized [22] that two distinct families of X_ℓ -Laguerre and X_ℓ -Jacobi polynomials exist, now labelled as type I and type II. The existence of precisely two families for each class was explained in [13] for the Laguerre case, by showing that these families are obtained from the classical ones by means of an isospectral algebraic Darboux transformation [9]. Indeed, there are four families of algebraic factorizations, only two of which give rise to isospectral transformations [10]. The equivalent result for the Jacobi case was shown in [14], and similar results on the relation between the Darboux transformation and exceptional polynomials have been published in [24].

Exceptional orthogonal polynomials and their associated exactly solvable potentials have found numerous recent applications in a range of problems in mathematical physics, such as mass-dependent potentials [18], Fokker-Planck and Dirac equations [16], supersymmetric quantum mechanics [5, 15] or quasi-exact solvability [26].

A natural question that arises from the previous works is whether the isospectral Darboux transformation can be iterated in a Darboux-Crum form [4, 3] in order to obtain families of exceptional orthogonal polynomials that enjoy spectral characterizations and completeness properties similar to the ones described above. Our purpose in this paper is to give a positive answer to this question by showing that *two-step* isospectral Darboux-Crum transformations can be used to construct a new set of exceptional orthogonal polynomials of Laguerre type. This family is indexed by two integer parameters (m_1, m_2) and a real parameter k . The sequence starts with degree $\ell = m_1 + m_2 - 1$ and it is complete in a weighted L^2 space endowed with a non-singular weight. Particular choices of m_1 and m_2 correspond to the type II X_ℓ -Laguerre and to the classical Laguerre polynomials.

Our paper is organized as follows. In Section 2, we define the two-step exceptional Laguerre polynomials, present their characterization as the unique polynomial solutions of a second-order linear differential equation (Theorem 1) and state their completeness properties (Theorem 2). We devote Section 3 to the proof of Theorem 1, using rational factorizations, an isospectral Darboux-Crum transformation and the higher-order intertwining relation (37). Section 4 addresses then the completeness property stated in Theorem 2 through a polynomial approximation argument for weighted L^2 spaces.

2. TWO-STEP EXCEPTIONAL LAGUERRE POLYNOMIALS

Our goal in this section is to introduce the two-step exceptional Laguerre polynomials, present their relation to second-order linear differential operators and state their completeness and orthogonality properties. The proofs of these results will be given in Sections 3 and 4.

We begin by recalling that for any given real number k , the classical n th degree Laguerre polynomial $L_n(z) = L_n^{(k)}(z)$ is the unique polynomial solution of the second-order differential equation

$$(1) \quad \mathcal{L}_k[L_n] = -nL_n,$$

where

$$(2) \quad \mathcal{L}_k[y] = zy'' + (k+1-z)y',$$

normalized by the condition

$$L_n(z) = \frac{(-z)^n}{n!} + \text{lower degree terms}.$$

For $k > -1$, the sequence $\{L_n^{(k)}\}_{n=0}^\infty$ spans a dense subspace of the Hilbert space

$$(3) \quad \mathcal{H}_k := L^2([0, \infty), W_k dz), \quad W_k(z) := z^k e^{-z},$$

and forms an orthogonal polynomial family satisfying

$$(4) \quad \int_0^\infty L_n^{(k)} L_j^{(k)} W_k(z) dz = \begin{cases} \frac{\Gamma(k+n+1)}{n!} & n = j \\ 0 & n \neq j. \end{cases}$$

The *two-step exceptional Laguerre polynomials* are now defined in the following way. Let k be real and $0 \leq m_1 < m_2$ integers. Set

$$(5) \quad \ell = m_1 + m_2 - 1$$

and define

$$(6) \quad \eta_{12} = \mathcal{W}[\eta_1, \eta_2], \text{ where } \eta_a = L_{m_a}^{(-k-2)}, \quad a = 1, 2,$$

and where \mathcal{W} denotes the Wronskian operator. Consider now the second-order linear differential operator $\hat{\mathcal{L}}$ defined by

$$(7) \quad \hat{\mathcal{L}}_k[y] = zy'' + (k+1-z)y' + \frac{2z}{\eta_{12}}(\eta'_{12}(y-y') + \mathcal{W}[\eta'_1, \eta'_2]y).$$

Let $n \geq \ell$ be an integer. Set

$$(8) \quad C := ((m_1 - m_2)(k+2-m_1+n)(k+2-m_2+n))^{-1}$$

and let \hat{L}_n be the polynomial defined by

$$(9) \quad \hat{L}_n := \hat{L}_n^{(k, m_1, m_2)} = Cz^{-k} \mathcal{W}[\eta_1, \eta_2, z^{k+2} L_{n-\ell}^{(k+2)}], \quad n \geq \ell$$

$$(10) \quad = \frac{(-z)^n}{(n-\ell)! m_1! m_2!} + \text{lower degree terms.}$$

For a lighter notation, we shall use \hat{L}_n instead of $\hat{L}_n^{(k, m_1, m_2)}$ throughout the paper. Whenever the symbol \hat{L}_n is used on its own, the rest of the parameters $k \in \mathbb{R}$, $m_1, m_2 \in \mathbb{N}$ are understood.

The main results of our paper are the following.

Theorem 1. *The above-defined \hat{L}_n is the unique polynomial solution of the differential equation*

$$(11) \quad \hat{\mathcal{L}}_k[\hat{L}_n] = (2\ell - n)\hat{L}_n, \quad n \geq \ell$$

subject to the normalization (10).

Theorem 2. *If $k > m_2 - 2$, then $\eta_{12}(z) > 0$ for $z \geq 0$. Furthermore, the above defined polynomials $\{\hat{L}_n\}_{n=\ell}^\infty$ span a dense subspace of the Hilbert space*

$$(12) \quad \mathcal{H}_{k, m_1, m_2} := L^2([0, \infty), \hat{W}_{k, m_1, m_2} dz), \quad \hat{W}_{k, m_1, m_2}(z) := z^k \eta_{12}^{-2} e^{-z}$$

and satisfy the following orthogonality relations:

$$(13) \quad \int_0^\infty \hat{L}_n \hat{L}_j \hat{W}_{k, m_1, m_2} dz = \begin{cases} \frac{C \Gamma(k+n+3-\ell)}{(m_1 - m_2)(n-\ell)!} & n = j \\ 0 & n \neq j. \end{cases}$$

If $m_1 = 0, m_2 = 1$, then by well-known identities for the Laguerre polynomials

$$\hat{L}_n^{(k, 0, 1)}(z) = \frac{z^2 L_n^{(k+2)''}(z) + (k+2)(2z L_n^{(k+2)'}(z) + (k+1)L_n^{(k+2)}(z))}{(k+2+n)(k+1+n)} = L_n^{(k)}(z).$$

Therefore, the above-defined polynomials generalize the classical ones. If $m_1 = 0, m_2 = m+1$, where $m \geq 1$, then

$$\frac{C}{k+2+n-m} \hat{L}_n^{(k, 0, m+1)} = z^{-k} \mathcal{W}[L_m^{(-k-1)}, z^{k+1} L_n^{(k+1)}] = L_{k, m, n}^{\text{II}}, \quad n \geq m,$$

where the latter are the recently introduced exceptional Laguerre polynomials of type II [21, 13]. Therefore, the newly defined polynomials \hat{L}_n also generalize and

extend the previously described class of exceptional Laguerre polynomials. This is natural since for $m_1 = 0$ one of the factorizing functions becomes a constant and therefore we are dealing with a 1-step Darboux transformation. The two families of X_ℓ -Laguerre exceptional polynomials of arbitrary codimension were shown to be obtainable by means of a 1-step Darboux transformation from the classical ones [13].

3. RATIONAL FACTORIZATIONS

This Section is devoted to the proof of Theorem 1, characterizing the two-step exceptional Laguerre polynomials \hat{L}_n as the unique normalized polynomial solutions of the differential equation (11).

Let

$$T[y] = p(z)y'' + q(z)y' + r(z)y$$

be a second-order differential operator with rational coefficients $p(z)$, $q(z)$ and $r(z)$.

Definition 1. A rational factorization of $T[y]$ is an operator identity of the form

$$T = BA + \tilde{\lambda}$$

where $A[y]$, $B[y]$ are first-order differential operators with rational coefficients.

We call

$$\hat{T} = AB + \tilde{\lambda}$$

the *partner operator* corresponding to the above rational factorization. Note that the following formal intertwining relations therefore hold by construction:

$$(14) \quad AT = \hat{T}A, \quad TB = B\hat{T}.$$

Given a rational factorization, let $\phi(z)$ be the unique (up to constant multiple) solution of $A[\phi] = 0$. Observe that $w(z) := \phi'(z)/\phi(z)$ is rational and that $T[\phi] = \tilde{\lambda}\phi$. For this reason we will call $\phi(z)$ a *quasi-rational* factorization function. Also note that the following identities hold for A and B :

$$(15) \quad A[y] = \frac{b(z)}{\phi} \mathcal{W}[\phi, y] = b(z)(y' - w(z)y)$$

$$(16) \quad B[y] = \frac{\hat{b}(z)}{\hat{\phi}} \mathcal{W}[\hat{\phi}, y] = \hat{b}(z)(y' - \hat{w}(z)y),$$

where $b(z)$ is a rational function called the *factorization gauge* [13], and

$$\hat{b} = \frac{p}{b}, \quad \hat{w} = -w - \frac{q}{p} + \frac{b'}{b}, \quad \hat{\phi} = \exp\left(\int \hat{w} dz\right).$$

The partner operator has the form

$$(17) \quad \hat{T}[y] = p(z)y'' + \hat{q}(z)y' + \hat{r}(z)y$$

where

$$(18) \quad \hat{q} = q + p' - \frac{2pb'}{b}, \quad \hat{r} = -p(\hat{w}' + \hat{w}^2) - \hat{q}\hat{w} + \tilde{\lambda}.$$

Let us also define measures Wdz and $\hat{W}dz$ by setting

$$W = p^{-1} \exp\left(\int \frac{q}{p} dz\right), \quad \hat{W} = \frac{\hat{b}}{b} W.$$

The above defined operators A and $-B$ are formally adjoint of each other with respect to these measures, in the sense that:

$$(19) \quad \int A[f]g \hat{W} dz + \int B[g]f W dz = \hat{b}W fg.$$

It is important to note that a rational factorization is just special case of the well-known Darboux transformation for Schrödinger operators, albeit expressed relative to a general coordinate and gauge. Indeed, the operator $T[\phi]$ can be related to a Schrödinger operator

$$(20) \quad H[\psi] = -\psi'' + U(x)\psi$$

by means of a change of variable $z = \zeta(x)$ satisfying

$$(21) \quad p(\zeta(x)) = \zeta'(x)^2$$

and a gauge transformation

$$(22) \quad -(\mu T[\phi]) \circ \zeta = H[\psi],$$

$$(23) \quad \psi = (\mu\phi) \circ \zeta,$$

$$(24) \quad \mu = \exp \int \frac{2q - p'}{4p} dz,$$

$$(25) \quad U = -(r \circ \zeta) + \frac{(\mu \circ \zeta)''}{(\mu \circ \zeta)}$$

In the physical variable and gauge, the above factorization corresponds to

$$H = (\partial_x + u)(-\partial_x + u) - \tilde{\lambda}, \quad u(x) = \frac{\psi'(x)}{\psi(x)},$$

and is related to the factorization function $\phi(z)$ by (22). The partner operator in the physical gauge is given by

$$(26) \quad \hat{H} = (-\partial_x + u)(\partial_x + u) - \tilde{\lambda} = -\partial_{xx} + \hat{U}$$

where the partner potential is given by the well-known Crum potential formula

$$(27) \quad \hat{U} = U - 2\partial_{xx} \log \psi.$$

Just as the Darboux transformation can be iterated to obtain the more general Darboux-Crum transformation, rational factorizations can be iterated and give rise to higher order intertwining relations. Indeed, we shall see that the operator $\hat{L}[y]$ defined in (7) is obtained from the classical Laguerre operator $L[y]$ by means of a 2-step factorization. We describe this construction in more detail, and thereby furnish a proof of Proposition 1.

Fix k, m_1, m_2 as above and let

$$(28) \quad T_0 = \mathcal{L}_{k+2}$$

be the ordinary Laguerre operator. The quasi-rational functions

$$\phi_i := z^{-k-2}\eta_i, \quad i = 1, 2$$

are factorization functions of T_0 with eigenvalues $\tilde{\lambda}_i = k + 2 - m_i$ [6, Section 6.1]. Consider the rational factorization of T_0 corresponding to the factorizing function ϕ_1 and factorizing gauge

$$(29) \quad b_1 = z\eta_1$$

which gives

$$(30) \quad T_0 = B_1 A_1 + \tilde{\lambda}_1,$$

where by equations (15) and (16) it is clear that

$$(31) \quad A_1[y] = \frac{b_1}{\phi_1} \mathcal{W}[\phi_1, y], \quad B_1[y] = \frac{y' - y}{\eta_1}.$$

Let

$$(32) \quad T_1 := \hat{T}_0 = A_1 B_1 + \tilde{\lambda}_1$$

be the corresponding partner operator. Note now that

$$\phi_{12} := A_1[\phi_2] = z^{k+3} \mathcal{W}[\phi_1, \phi_2] = z^{-k-1} \eta_{12}$$

is a quasi-rational factorization function of T_1 with eigenvalue $\tilde{\lambda}_2 = k+2-m_2$. This means that ϕ_{12} can be used as factorization function for a rational factorization of T_1 , together with the factorization gauge

$$(33) \quad b_{12} = \frac{z\eta_{12}}{\eta_1}.$$

We thus obtain the factorization

$$(34) \quad T_1 = B_2 A_2 + \tilde{\lambda}_2$$

where

$$A_2[y] = \frac{b_{12}}{\phi_{12}} \mathcal{W}[\phi_{12}, y].$$

We call T_2 the partner operator of this last factorization

$$(35) \quad T_2 := \hat{T}_1 = A_2 B_2 + \tilde{\lambda}_2$$

and we observe that

$$(36) \quad (A_2 A_1)[y] = b_{12} b_1 \frac{\mathcal{W}[\phi_1, \phi_2, y]}{\mathcal{W}[\phi_1, \phi_2]} = z^{-k} \mathcal{W}[\eta_1, \eta_2, z^{k+2} y].$$

By construction, the following higher-order intertwining relation holds

$$(37) \quad T_2 A_2 A_1 = A_2 A_1 T_0.$$

From the definition of the \hat{L}_n polynomials in (9) and the above expression (36) we see that \hat{L}_n are eigenpolynomials of the differential operator T_2 :

$$(38) \quad T_2[\hat{L}_n] = (\ell - n)\hat{L}_n.$$

To conclude the proof of Theorem 1, we must establish that T_2 and $\hat{\mathcal{L}}_k$ are related by

$$(39) \quad \hat{\mathcal{L}}_k = T_2 + \ell.$$

Let q_i and r_i , $i = 0, 1, 2$ be the coefficients of operators T_i above:

$$(40) \quad T_i[y] = zy'' + q_i y' + r_i y, \quad i = 0, 1, 2.$$

From (2) and (28), we see for instance that

$$(41) \quad q_0 = k + 3 - z,$$

$$(42) \quad r_0 = 0$$

and hence by (18) (29) (33),

$$\begin{aligned}
 (43) \quad q_2 &= q_1 + p' - 2p \frac{b'_{12}}{b_{12}} \\
 &= q_0 + 2p' - 2p \log(b_1 b_{12})' \\
 &= 1 + k - z - 2z \frac{\eta'_{12}}{\eta_{12}}.
 \end{aligned}$$

which matches the first order part of (7). It remains to demonstrate that

$$(44) \quad r_2 = \frac{2z}{\eta_{12}}(\eta'_{12} + \mathcal{W}[\eta'_1, \eta'_2]) - \ell,$$

and for this purpose it will be useful to compare these expressions in the Schrödinger gauge by making use of the Crum potential formula.

Let $U_i(x), \mu_i(z)$, $i = 0, 1, 2$ be the potentials and gauge factors corresponding to the operators T_i , as given by (24) -(25). In particular, by (43),

$$(45) \quad \mu_0 = e^{-\frac{z}{2}} z^{\frac{5}{4} + \frac{k}{2}},$$

$$(46) \quad \mu_2 = \frac{\mu_0}{z\eta_{12}} = \frac{e^{-\frac{z}{2}} z^{\frac{k}{2} + \frac{1}{4}}}{\eta_{12}}$$

where the latter follows by (43). Since $p(z) = z$, the required change of variables is

$$(47) \quad \zeta(x) = \frac{x^2}{4},$$

and by the Crum potential formula (27), we have

$$(48) \quad U_2 = U_0 - 2\partial_{xx} \log \mathcal{W}[\psi_1, \psi_2],$$

where $\psi_i = (\mu_0 \phi_i) \circ \zeta$. It follows that

$$\begin{aligned}
 (49) \quad \log \mathcal{W}[\psi_1, \psi_2] &= (\log \mathcal{W}[\phi_1, \phi_2] + \frac{1}{2} \log p + 2 \log \mu_0) \circ \zeta \\
 &= \log(\eta_{12} \circ \zeta) - (1 + k) \log \zeta - \zeta
 \end{aligned}$$

By (25), (48) and (42) we have

$$r_2 \circ \zeta = 2(\log \mathcal{W}[\psi_1, \psi_2])'' + \frac{(\mu_2 \circ \zeta)''}{(\mu_2 \circ \zeta)}(x) - \frac{(\mu_0 \circ \zeta)''}{(\mu_0 \circ \zeta)}(x).$$

Applying the identity

$$f''/f - g''/g = (\log(f/g))'' + (\log(f/g))'(\log(fg))'$$

with $f = \mu_2 \circ \zeta$ and $g = \mu_0 \circ \zeta$ and relation (46) gives

$$\begin{aligned}
 r_2 \circ \zeta &= (\log(\eta_{12} \circ \zeta) - 2\zeta - (2k + 3) \log \zeta)'' + \\
 &\quad + (\log \zeta + \log(\eta_{12} \circ \zeta))'(\log \zeta + \log(\eta_{12} \circ \zeta) - 2 \log(\mu_0 \circ \zeta))' \\
 &= \frac{(\eta_{12} \circ \zeta)''}{(\eta_{12} \circ \zeta)} - (2\zeta + (2k + 3) \log \zeta)'' + (\log \zeta)'(\log \zeta - 2 \log(\mu_0 \circ \zeta))' \\
 &\quad + 2(\log(\eta_{12} \circ \zeta))'(\log \zeta - (\mu_0 \circ \zeta))' \\
 &= \frac{(\eta_{12} \circ \zeta)''}{(\eta_{12} \circ \zeta)} + 2(\log(\eta_{12} \circ \zeta))'(\log \zeta - (\mu_0 \circ \zeta))'
 \end{aligned}$$

Now switching back to the algebraic variable gives

$$\begin{aligned}
 r_2 &= \frac{(z\eta''_{12} + \frac{1}{2}\eta'_{12})}{\eta_{12}} + 2z \left(\frac{1}{z} - \frac{\mu'_0}{\mu_0} \right) \frac{\eta'_{12}}{\eta_{12}} \\
 (50) \quad &= \frac{z\eta''_{12}}{\eta_{12}} + (z - k) \frac{\eta'_{12}}{\eta_{12}}.
 \end{aligned}$$

Since $y = \eta_i$, $i = 1, 2$ are solutions of the Laguerre differential equation,

$$(51) \quad zy'' - (1 + k + z)y' + m_i y = 0,$$

we have

$$z\eta''_{12} - (k + z)\eta'_{12} - 2z\mathcal{W}[\eta'_1, \eta'_2] + \ell\eta_{12} = 0,$$

which together with (50) implies (44), the required conclusion. We have thus established that the polynomials \hat{L}_n defined by (9) are obtained from the associated Laguerre polynomials $L_n^{(k)}$ by use of a two-step rational factorization. The proof of Theorem 1 is achieved by showing that the differential equation $\hat{\mathcal{L}}_k$ in (7) satisfied by the polynomials \hat{L}_n is essentially the partner of Laguerre's operator, corresponding to the iterated Darboux transformation. It is important to stress that the two factorizing functions of the iterated rational factorization:

$$\phi_1 = z^{-k-2} L_{m_1}^{(-k-2)}(z), \quad \phi_2 = z^{-k-2} L_{m_2}^{(-k-2)}(z)$$

are indexed by two integers m_1 and m_2 that can be freely chosen. This fact should be put in comparison with the restricted choice of factorizing functions in the original Crum method [3] or its modification by Adler [1]. The main difference lies also in the fact that in the former constructions the Darboux transformations are state-deleting, while in this paper the factorizing functions correspond to isospectral transformations.

4. PROOFS OF COMPLETENESS AND ORTHOGONALITY PROPERTIES

Lemma 1. *Let $m_2 > m_1 \geq 0$ be integers and $k > m_2 - 2$ a real number. Then, $\eta_{12}(z) > 0$ for $z \geq 0$.*

Proof. Since $y = \eta_i$, $i = 1, 2$ satisfies (51), η_{12} satisfies the inhomogeneous first-order differential equation

$$z\eta'_{12} - (k + 1 + z)\eta_{12} + (m_2 - m_1)\eta_1\eta_2 = 0.$$

This gives

$$\eta_{12}(z) = (m_2 - m_1)e^z z^{1+k} \int_z^\infty e^{-t} t^{-2-k} \eta_1(t) \eta_2(t) dt.$$

If $k > m - 2$, then by Theorem 6.73 of [25], $L_m^{(-k-2)}(z) > 0$ for $z \geq 0$. The desired conclusion follows immediately.

Let \mathcal{P} denote the vector space of univariate polynomials, and \mathcal{P}_n the subspace of polynomials of degree $\leq n$. Let $\ell = m_1 + m_2 - 1$ and let $\mathcal{E}_n \subset \mathcal{P}_{n+\ell}$ denote the span of $\hat{L}_\ell, \dots, \hat{L}_{n+\ell}$. Let z_i , $i = 1, \dots, \ell$ denote the zeroes of $\eta_{12}(z)$.

Lemma 2. *A polynomial $y \in \mathcal{P}_{n+\ell}$ belongs to \mathcal{E}_n if and only if*

$$(52) \quad -2z_i y'(z_i) + \left(z_i - k + \frac{z_i \eta''_{12}(z_i)}{\eta'_{12}(z_i)} \right) y(z_i) = 0, \quad i = 1, \dots, \ell$$

Proof. Suppose that $y \in \mathcal{E}_n$. By equations (43)–(50) of the previous section,

$$\operatorname{Res} \left\{ (-2zy' \frac{\eta'_{12}}{\eta_{12}} + y \frac{(z-k)\eta'_{12} + z\eta''_{12}}{\eta_{12}}, z = z_i \right\} = 0, \quad i = 1, \dots, \ell.$$

These are precisely the relations (52). Since the codimension of \mathcal{E}_n in $\mathcal{P}_{n+\ell}$ is exactly ℓ , the constraints (52) define the subspace \mathcal{E}_n , as claimed.

Lemma 3. *Let $\eta(z)$ be a polynomial such that $\eta(z) \neq 0$ for $z \geq 0$. If $k > -1$, then the subspace $\eta\mathcal{P} := \{\eta(z)p(z) \mid p \in \mathcal{P}\}$ is dense in \mathcal{H}_k .*

Proof. Since \mathcal{P} is dense in \mathcal{H}_k [25], it suffices to prove that \mathcal{P} is contained in the L^2 closure of $\eta\mathcal{P}$. Let $p \in \mathcal{P}$ and $\epsilon > 0$ be given. We seek a polynomial $q \in \mathcal{P}$ such that $\|p - q\eta\|_k \leq \epsilon$. To this end let $z_0 = 1 + \max\{|z| : \eta(z) = 0\}$ and set

$$r(z) = \begin{cases} \frac{p(z-z_0)}{\eta(z-z_0)} & z \geq z_0, \\ 0 & \text{otherwise} \end{cases}$$

Set $l = k + 2 \deg \eta$ and observe that

$$z^k |\eta(z)|^2 < (z + z_0)^l, \quad z \geq 0.$$

Since $1/|\eta(z)|$ is bounded for $z \geq 0$, we have

$$\|r\|_l = \int_{z_0}^{\infty} \left| \frac{p(z-z_0)}{\eta(z-z_0)} \right|^2 z^l e^{-z} dz = e^{-z_0} \int_0^{\infty} \left| \frac{p(z)}{\eta(z)} \right|^2 (z + z_0)^l e^{-z} dz < \infty$$

Since \mathcal{P} is dense in \mathcal{H}_ℓ , there exists a $\tilde{q} \in \mathcal{P}$ such that $\|r - \tilde{q}\|_l \leq \epsilon$. Hence, $q(z) = \tilde{q}(z + z_0)$ is the desired polynomial because

$$\begin{aligned} \|p - q\eta\|_k &= \int_0^{\infty} |r(z + z_0) - \tilde{q}(z + z_0)|^2 |\eta(z)|^2 z^k e^{-z} dz \\ &< \int_0^{\infty} |r(z + z_0) - \tilde{q}(z + z_0)|^2 |z + z_0|^l e^{-z} dz \\ &< e^{z_0} \int_0^{\infty} |r(z) - \tilde{q}(z)|^2 z^l e^{-z} dz \\ &< \epsilon \end{aligned}$$

This concludes the proof of Lemma 3.

We now use Lemma 2 and Lemma 3 to prove Theorem 2. The positivity of $\eta_{12}(z)$ was established in Lemma 1. It therefore follows that the measure $\hat{W}_{k,m_1,m_2}(z)dz$ has finite moments. Let us now prove that the set of all 2-step Darboux exceptional Laguerre polynomials

$$\mathcal{E} = \bigcup_n \mathcal{E}_n = \operatorname{span}\{\hat{L}_n\}_{n=\ell}^{\infty}$$

is dense in \mathcal{H}_{k,m_1,m_2} . Let $f \in \mathcal{H}_{k,m_1,m_2}$ and $\epsilon > 0$ be given. Set $\tilde{f} = f/\eta_{12}$ and note that $\tilde{f} \in \mathcal{H}_k$ because $\|f\|_{k,m_1,m_2} = \|\tilde{f}\|_k$. Hence, by Lemma 3, there exists a polynomial $p \in \mathcal{P}$ such that

$$\|f - p\eta_{12}^2\|_{k,m_1,m_2} = \int_0^{\infty} |\tilde{f}(z) - \eta_{12}(z)p(z)|^2 z^k e^{-z} dz \leq \epsilon.$$

By Lemma 2, $p\eta_{12}^2 \in \mathcal{E}$, which establishes the claim.

To conclude the proof of Theorem 2 let us prove the orthogonality (13) of the exceptional polynomials. Let $A_i[y], B_i[y]$, $i = 1, 2$ be the first-order operators defined in the preceding section. For $f, g \in \mathcal{P}$, by (14)(19) (30) (34)

$$\begin{aligned} \langle A_2 A_1 f, A_2 A_1 g \rangle_{k, m_1, m_2} &= \langle B_1 B_2 A_2 A_1 f, g \rangle_{k+2} \\ &= \left\langle B_1 (T_1 + \tilde{\lambda}_2) A_1 f, g \right\rangle_{k+2} \\ &= \left\langle (T_0 + \tilde{\lambda}_2)(T_0 + \tilde{\lambda}_1) f, g \right\rangle_{k+2} \end{aligned}$$

The desired conclusions follow by (1) and (4).

5. SUMMARY AND CONCLUSIONS

We have shown how the iterated Darboux or Darboux-Crum transformation can be used to construct new families of exceptional orthogonal polynomials, thus enlarging the class previously described in the literature. Due to the many properties in common with classical orthogonal polynomials, a systematic classification of all the exceptional families is an important goal, and the Darboux-Crum construction described in this paper seems an appropriate tool to achieve it.

In the Schrödinger gauge and physical variable, this construction gives rise to new exactly solvable potentials in quantum mechanics.

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